Slide 1

Session 6
Factorials, Combinations, Summations and Products

Slide 2

Factorials

A factorial (math symbol !) is shorthand for “multiply this positive integer by every positive integer less than it”.

$$\begin{aligned} 3! &= 3 \cdot 2 \cdot 1 = 6 \\ 5! &= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5 \cdot 4 \cdot 3! = 120 \\ 1! &= 1 \\ 0! &= 1 \text{ (by definition)} \\ n! &= n[(n-1)!] \end{aligned}$$

Try 70! on a calculator. What happens?

Slide 3

Factorials

In fact, n! measures “How many different ways can I order n distinct items?” Consider $n = 3$, items A, B, and C.

ABC
ACB
BAC
BCA
CAB
CBA

$3! = 6$

Why does this work?

Slide 4

Combinations

Related to the factorial, a combination $\binom{n}{x}$ is shorthand for “n choose x”, which means, “How many ways can I choose x objects from among n without regard to order?”

Example: You give 5 patients a treatment and 3 of them get better. How many ways could that happen?

Call the patients A, B, C, D, and E. The patients who get better could be any of these combinations (the order is not important):

ABC	ACD	BCD	CDE	ABD
ABE	ADE	BDE	ACE	BCE

Slide 5

Combinations

So, there are 10 ways to choose 3 patients from among 5.

$$\binom{5}{3} = 10$$

Slide 6

Combinations

We can compute combinations like this:

$$\binom{n}{x} = \frac{n!}{x!(n-x)!}$$

$$\binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{5!}{3!2!} = \frac{5 \cdot 4 \cdot 3!}{3!2!} = \frac{5 \cdot 4}{2 \cdot 1} = \frac{20}{2} = 10$$

The "nCr" button on your calculator does combinations. Find it and practice a few! Some textbooks also use this "nCr" notation.

Slide 7

Rules for Combinations

$$\binom{n}{0} = 1$$

$$\binom{n}{n} = 1$$

$$\binom{n}{1} = n$$

$$\binom{n}{n-1} = n$$

$$\binom{n}{x} = \binom{n}{n-x}$$

Slide 8

Practice with Combinations

$$\binom{4}{3} =$$

$$\binom{3}{1} =$$

$$\binom{2}{2} =$$

$$\binom{6}{3} =$$

Slide 9

Summation

The summation symbol, $\sum$ ("sigma"), is shorthand for "add up a bunch of numbers".

$$\sum_{i=1}^{30} (\qquad)$$

$i = 1$: This means start with the first element

30: This means continue through the 30th element

Inside the ( ) will be a mathematical expression, usually with an "i" index.

Slide 10

Summation

Suppose we have this expression:

$$\sum_{i=1}^{10} x_i$$

and we have 10 numbers in a list, x. We can assign the index i to represent each number's position in the list. So $x_1 = 8$, $x_2 = 3$, etc.

i	1	2	3	4	5	6	7	8	9	10
x	8	3	10	2	1	4	2	2	3	2

Slide 11

Summation

$$\begin{aligned}\sum_{i=1}^{10} x_i &= x_1 + x_2 + x_3 + \ldots + x_{10}\\ &= 8 + 3 + 10 + \ldots + 2\\ &= 37\end{aligned}$$

Slide 12

Summation: Examples in Statistics

Summation is a convenient abbreviation seen often in statistics.

Here's the formula for the mean (average):

$$\overline{x} = \sum_{i=1}^{n} \frac{x_i}{n} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

where x_i are the items to average and n is the number of items

Slide 13

Summation: Examples in Statistics

Here's the formula we saw online for calculating the standard deviation:

1. Compute the average
2. Subtract the average from each individual value
3. Square the differences
4. Sum the squared differences
5. Divide by n-1
6. Compute the square root

Slide 14

Summation: Examples in Statistics

Here's the formula in summation notation:

$$\text{SD} = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n-1}}$$

Slide 15

Summation: Examples in Statistics

Here's the formula we learned for the χ^2 statistic:

$$\chi^2 = \sum_{i=1}^{4} \frac{(|O_i - E_i| - 0.5)^2}{E_i}$$

Slide 16

Product

The product symbol, Π ("pi"), is shorthand for "multiply a bunch of numbers".

Suppose we have the same list of numbers as before, and the expression:

$$\prod_{i=1}^{10} x_i$$

Then

$$\prod_{i=1}^{10} x_i = 8 \cdot 3 \cdot 10 \cdot 2 \cdot 1 \cdot 4 \cdot 2 \cdot 2 \cdot 3 \cdot 2 = 46,080$$

Slide 17

Useful Properties of Summation

Factors inside a summation that are not indexed can often be taken outside the summation. For example:

$$\sum_{i=1}^{n} ax_i = a\sum_{i=1}^{n} x_i$$

$$\begin{aligned}\sum_{i=1}^{n}(ax_i + by_i + cz_i) &= \sum_{i=1}^{n} ax_i + \sum_{i=1}^{n} by_i + \sum_{i=1}^{n} cz_i \\ &= a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} y_i + c\sum_{i=1}^{n} z_i\end{aligned}$$

$$\sum_{i=1}^{n} a = na$$

So, $\sum(a + x_i) = \sum a + \sum x_i = na + \sum x_i$

Slide 18

Summation Warnings!

However:

$$\sum_{i=1}^{n} x_i y_i \neq \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)$$

$$\sum_{i=1}^{n} (x_i)^2 \neq \left(\sum_{i=1}^{n} x_i\right)^2$$

Slide 19

Example: Summation of Combinations

$$\begin{aligned}\sum_{i=0}^{3}\binom{4}{i} &= \binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} \\ &= 1 + 4 + 6 + 4 \\ &= 15\end{aligned}$$

Slide 20

Example: Summation of Logarithms

$$\begin{aligned}\sum_{j=1}^{3}\ln(j) &= \ln(1) + \ln(2) + \ln(3) \\ &= 0 + 0.693 + 1.099 \\ &= 1.792\end{aligned}$$

Slide 21

Practice with Summations

Suppose you have the following lists x, y, and z and variables a, b and c which have the values shown.

i	x	y	z	
1	1	4	7	a = 2
2	2	5	8	b = 3
3	3	6	9	c = 4
				n = ?

Compute:

$$\sum_{i=1}^{n} a x_i \qquad a\sum_{i=1}^{n} x_i$$

$$\sum_{i=1}^{n} a \qquad na$$

Slide 22

Practice with Summations

$$\sum_{i=1}^{n}(ax_i + by_i + cz_i) \qquad a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} y_i + c\sum_{i=1}^{n} z_i$$

$$\sum_{i=1}^{n} x_i y_i \qquad \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)$$

$$\sum_{i=1}^{n} (x_i)^2 \qquad \left(\sum_{i=1}^{n} x_i\right)^2$$

Slide 23

Example Using Combinations

You are designing a Phase I clinical trial to test a new chemotherapeutic agent. The goal of a Phase I study is to find the highest dose that can be safely administered to a patient. You start by giving a low dose to 3 patients and monitoring them for toxicity. If any of the patients experience toxicity, you will stop. If none of the patients experience toxicity, you will try a higher dose on 3 more patients. If the true underlying toxicity rate at the low dose is 5%, what is the chance the trial will escalate to the next higher dose?

Slide 24

Example Using Combinations

You will soon learn about the binomial distribution which gives

$$\begin{aligned} \text{p(escalate dose)} &= \text{p(no patients out of 3 at the low dose experience toxicity)} \\ &= \binom{3}{0}(0.05)^0(0.95)^3 \\ &= 1(1)(0.857375) \\ &= 0.857 \end{aligned}$$

The probability of escalating to a higher dose is 0.857, or 85.7%.

Slide 25

Example Using Combinations

What if you had 5 patients and decided to stop if there were any toxicities?

Slide 26

Example using Combinations

What if you had 5 patients and decided to stop if there were 2 or more toxicities?

Slide 27

Solutions to Practice Problems

Slide 8:

$$\binom{4}{3} = 4$$

$$\binom{3}{1} = 3$$

$$\binom{2}{2} = 1$$

$$\binom{6}{3} = 20$$

Slide 28

Solutions to Practice Problems

Slide 21:

n = 3

$$\sum_{i=1}^{n} ax_i = 12 \qquad a\sum_{i=1}^{n} x_i = 12$$

$$\sum_{i=1}^{n} a = 6 \qquad na = 6$$

Slide 29

Solutions to Practice Problems

Slide 22:

$$\sum_{i=1}^{n}(ax_i + by_i + cz_i) = 153 \qquad a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} y_i + c\sum_{i=1}^{n} z_i = 153$$

$$\sum_{i=1}^{n} x_i y_i = 32 \qquad \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right) = 90$$

$$\sum_{i=1}^{n}(x_i)^2 = 14 \qquad \left(\sum_{i=1}^{n} x_i\right)^2 = 36$$