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Session 4

Logarithms and Exponentials

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Logarithms

A **logarithm** is a property of a number related to its exponents. (A logarithm is the *mathematical inverse* of an exponent.)

If $a^c = b$, then c is the logarithm, with base a, of b. In math terms:

$$c = \log_a b$$

Alternatively, c is the power to which we raise a to get b. Since $5^3 = 125$, then $3 = \log_5 125$.

The base you will use most often is a special number, e. $e = 2.71828182846$ and logs using base e are called natural logs (abbreviated "ln").

If someone just writes "log", it is important to verify that they mean natural log (ln) and not $\log_{10}$.

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Properties of Logarithms

- If you have numbers ranging from 0 to ∞, the logarithms of those numbers will range from $-\infty$ to $+\infty$
- Logarithms are undefined for negative numbers
- The logarithm of 0 is $-\infty$
- The logarithm of 1 is 0 (any base)
- $\ln(a \times b) = \ln a + \ln b$
- $\ln(b \times b) = \ln b + \ln b = 2 \ln b$
- $\ln(a^b) = b \times \ln a$
- $\ln(\frac{a}{b}) = \ln a - \ln b$
- **Beware!** $ln(a + b) \neq ln(a) + ln(b)$
- **Beware!** $ln(a - b) \neq ln(a) - ln(b)$

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Logarithms on Your Calculator

Locate these buttons:

- LOG: This is $\log_{10}$ (log base 10, or the *common* log)
- LN: This is the natural log
- e or e^x (sometimes written *exp* or *exp*(x)): This is the base of the natural log

Try these on your calculator:

$\ln 0$	$\ln(3 \times 5) = \ln 3 + \ln 5$
$\ln 0.000001$	
$\ln 1$	$\ln(3^2) = 2 \ln 3$
$\ln 1.1$	
$\ln 5$	$\ln(\frac{3}{5}) = \ln(3) - \ln(5)$
$\ln 10$	$\log 10$

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ln and e: Inverse Functions

If you take the natural logs of numbers, you can exponentiate them using e to return them to their original value ($e^{\ln x} = x$):

x	$\ln x$	$e^{\ln x}$
1	0	1
1.5	0.40546	1.5
3	1.0986	3

Functions like these are called *inverse functions*.

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Log Examples

x	Common log (base 10)	Natural log (base e)	
1	$log_{10}(1) = 0$	$log_e(1) = ln(1) = 0$	$[e^0 = 1]$
10	$log_{10}(10) = 1$	$ln(10) = 2.3$	$[e^{2.3} = 10]$
100	$log_{10}(100) = 2$	$ln(100) = 4.6$	$[e^{4.6} = 100]$
1,000	$log_{10}(1000) = 3$	$ln(1000) = 6.9$	$[e^{6.9} = 1000]$
10,000	$log_{10}(10000) = 4$	$ln(10000) = 9.2$	$[e^{9.2} = 10000]$

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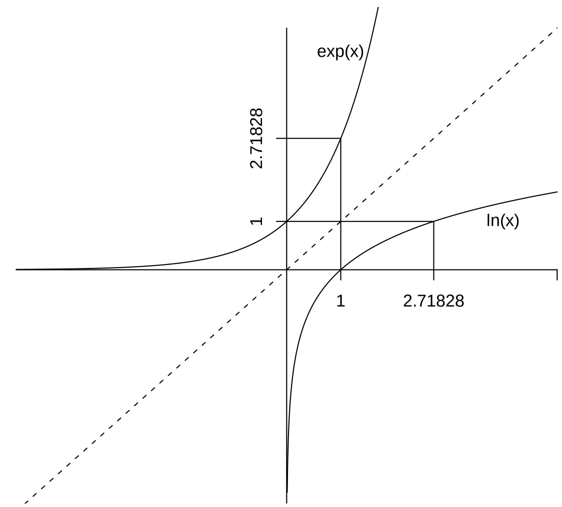

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Exponentials: Example 1

The normal distribution is commonly used in statistics to describe continuous data that are approximately bell-shaped. Let's graph the density function of the standard normal distribution (having mean 0 and standard deviation 1), namely

$$y = \frac{1}{\sqrt{2\pi}} exp(-x^2/2)$$

where $\pi = 3.141592654...$

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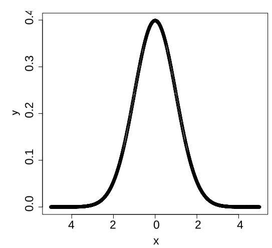
Exponentials: Example 1

$$y = \frac{1}{\sqrt{2\pi}} exp(-x^2/2)$$

x	y
0	0.399
± 0.5	0.352
± 1	0.242
± 1.5	0.130
± 2	0.054



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Exponentials: Example 1

For what value (or values) of x does $y = 0.1$?

$$\begin{aligned}
0.1 &= \frac{1}{\sqrt{2\pi}} exp(-x^2/2) \\
(0.1)(\sqrt{2\pi}) &= exp(-x^2/2) \\
0.2507 &= exp(-x^2/2) \\
\ln(0.2507) &= -x^2/2 \\
-1.3836 &= -x^2/2 \\
2(1.3836) &= x^2 \\
2.767 &= x^2 \Rightarrow x = \pm\sqrt{2.767} = \pm 1.66
\end{aligned}$$

Now go back to check your solutions and compare with our graph.

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Exponentials: Example 2

Consider a scenario where a units of a bacteria are put on a petri dish having unlimited food and resources. Assume that the bacteria double every 10 hours. Then a model for the total number of units of bacteria at any point in time t may be given by

$$y = ae^{bt}$$

Determine the value of b.

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Exponentials: Example 2

At time $t = 0$, $y = a$:

$$a = ae^{b(0)} = ae^0 = a(1) = a$$

Correct!

At time $t = 10$, $y = 2a$:

$$\begin{aligned}
2a &= ae^{b(10)} \\
2 &= e^{10b} \\
\ln(2) &= 10b \Rightarrow b = (\ln(2))/10 = 0.0693
\end{aligned}$$

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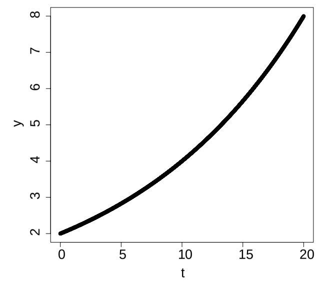
Exponentials: Example 2

Suppose $a = 2$ units. Graph the function $y = 2e^{0.0693t}$

t	y
0	2
5	2.83
10	4
15	5.66
20	8



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Exponentials: Example 2

How long does it take the bacteria to triple in quantity (from 2 to 6 units)?

$$\begin{aligned} 6 &= 2exp(0.0693t) \\ 3 &= exp(0.0693t) \\ \ln(3) &= 0.0693t \Rightarrow t = \ln(3)/0.0693 = 15.9 \text{ hours} \end{aligned}$$

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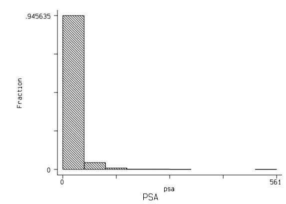
Logarithms: Example 1

Sometimes it's easier to do statistical analyses with values that have a bell-shaped distribution. And sometimes, the natural logarithms of numbers have a more bell-shaped distribution than the raw numbers themselves. Here is a graph of the PSA values for 1,913 men treated for prostate cancer:



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Logarithms: Example 1

Here are some of the numbers and their natural log transforms:

psa	ln(psa)
52	3.951244
97	4.574711
35.8	3.577948
24.8	3.210844
7.2	1.974081
63.4	4.149464
7.3	1.987874
7.9	2.066863
75	4.317488
22	3.091043
10	2.302585
13.4	2.595255

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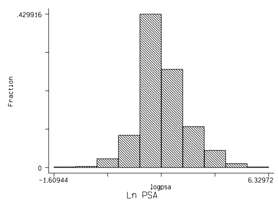
Logarithms: Example 1

Here is a graph of the natural log-transformed PSA values:



Some statistical procedures are more appropriate for data that are normally distributed, or bell shaped, similar to that above.

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Logarithms: Example 1

We can use e to transform the log numbers back to their originals.

psa	ln(psa)	exp(ln(psa))	
52	3.951244	52	
97	4.574711	96.99998	(slight roundoff errors)
35.8	3.577948	35.8	
24.8	3.210844	24.8	
7.2	1.974081	7.2	
63.4	4.149464	63.39999	
7.3	1.987874	7.3	
7.9	2.066863	7.900001	
75	4.317488	75.00001	
22	3.091043	22	
10	2.302585	10	
13.4	2.595255	13.4	

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Logarithms: Example 2

Recall from the online material the table of head injury and bicycle helmet use:

Wearing Helmet		No	Yes
Head	Yes	218	17
Injury	No	428	130

We can construct an ”odds ratio” to quantify the strength of the association between helmet use and head injury.

a	b
c	d

$$\widehat{OR} = \frac{ad}{bc}$$

An odds ratio of 1 implies that there is no association between helmet use and head injury.

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Logarithms: Example 2

Wearing Helmet		No	Yes
Head	Yes	218	17
Injury	No	428	130

$$\widehat{OR} = \frac{ad}{bc} = \frac{(218)(130)}{(17)(428)} = \frac{28340}{7276} = 3.9$$

People who don’t wear helmets have 3.9 times the odds of having a head injury if involved in an accident as people who wear helmets.

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Logarithms: Example 2

Suppose we want to put a confidence interval (CI) around the odds ratio. A confidence interval is a way of saying, "Given our data, the odds ratio might range between (some lower number) and (some higher number)." Since the odds ratio cannot go below 0, we use logarithms to assure that the confidence interval does not take on negative numbers. Here is the formula:

$$95\%\text{ CI for } \ln(\widehat{OR}) = \ln(\widehat{OR}) \pm 1.96 * sd[\ln(\widehat{OR})]$$

$\pm$ means "plus or minus" (2 calculations)

sd stands for standard deviation (we discussed earlier).

$$sd[\ln(\widehat{OR})] = \sqrt{\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}}$$

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Logarithms: Example 2

$$\begin{aligned}
\ln(\widehat{OR}) &= \ln 3.9 = 1.36 \\
sd[\ln(\widehat{OR})] &= \sqrt{\frac{1}{218} + \frac{1}{17} + \frac{1}{428} + \frac{1}{130}} \\
&= \sqrt{0.0046 + 0.0588 + 0.0023 + 0.0077} = 0.271 \\
95\%\text{ CI } \ln(\widehat{OR}) &= 1.36 - (1.96)(0.271), 1.36 + (1.96)(0.271) \\
95\%\text{ CI } \ln(\widehat{OR}) &= (0.83, 1.89) \text{ Be sure these bracket the } \ln(\widehat{OR}) \\
95\%\text{ CI } \widehat{OR} &= (e^{0.83}, e^{1.89}) \\
95\%\text{ CI } \widehat{OR} &= (2.29, 6.62) \text{ Be sure these bracket the } \widehat{OR}
\end{aligned}$$

We are 95% confident that the interval (2.29, 6.62) covers the true population odds ratio. You will see many more examples of odds ratios and confidence intervals in upcoming courses.

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Solutions to Practice Problems

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$\ln 0 =$ "Error"	$\ln(3 \times 5) = \ln 3 + \ln 5 = 2.71$
$\ln 0.000001 = -13.8$	
$\ln 1 = 0$	$\ln(3^2) = 2\ln 3 = 2.2$
$\ln 1.1 = 0.095$	
$\ln 5 = 1.61$	$\ln(\frac{3}{5}) = \ln 3 - \ln 5 = -0.51$