Foundations of Statistics I: Decision Theory
Problem Set 4

Problem 1 10 points. Consider the James-Stein estimator, and write a program to graph \( R(\theta, \delta_1)/p \) as a function of \( \frac{1}{p} \sum \theta_i^2 \). Choose your p wisely.

Problem 2 [20 points]
Consider a sample \( x = (x_1, x_2, \ldots, x_5) \) where each \( x_i \sim \text{Pois}(\lambda_i) \). Set \( x_2 = x_3 = 0 \) and fix \( x_4 \) and \( x_5 \) to your favorite positive integers. In everything that follows, \( x_1 \) will be allowed to vary and \( x_2, \ldots, x_5 \) will be fixed. You want to estimate \( \lambda_1 \) under squared error loss.

Let \( \delta_1 \) be the posterior mean of \( \lambda_1 \) assuming that \( \lambda_i \sim \text{Exp}(1), i = 1, \ldots, 5 \)
Let \( \delta_{EB} \) be the empirical Bayes estimate of the posterior mean of \( \lambda_1 \), assuming that \( \lambda_i \sim (1 - \alpha)I_0 + \alpha \text{Exp}(\gamma), \alpha \in (0, 1) \), where \( \gamma > 0 \) is the mean of the exponential.

Write computer programs to perform the following tasks:

1. Using an empirical Bayes approach, determine maximum likelihood estimates of \( \alpha \) and \( \gamma \). These will be functions of \( x \).
2. Graph \( \delta_{EB} \) and \( \delta_1 \) versus \( x_1 \); determine whether the relationship is linear or nonlinear; choose your resolution and range so that your conclusion is supported by the figure;
3. Graph the risk functions of \( \delta_{EB} \) and \( \delta_1 \), as you vary \( \lambda_1 \) and fix the other coordinates \( \lambda_i = x_i i \geq 2 \). These risk functions are averages over all possible values of \( x_1 \).

Problem 3 \( x \) is exponential with rate \( \theta \). Consider the class of estimators \( \delta_c(x) = cx \) and determine the best estimator of \( 1/\theta \) under squared error loss. Show that the estimator is a generalized Bayes estimator and discuss its admissibility.